

# Epistemic and aleatory uncertainty (Bayesian Approach)

Or what have a boxer , a wrestler and a coin in common?

---

## Introduction

We follow the illustration in [1].

Suppose we model a fair coin flip as random variable X as an example for *aleatory* uncertainty:

```
In[130]:= X = BernoulliDistribution[1/2]
```

```
Out[130]= BernoulliDistribution[1/2]
```

Additionally, we model the result of a fight world-best boxer against world-best wrestler as random variable Y, which has a parameter  $\theta \in \{0,1\}$  which in turn is uniformly distributed:

```
In[131]:= θ = UniformDistribution[{0, 1}];  
Y = BernoulliDistribution[θ]
```

```
Out[132]= BernoulliDistribution[UniformDistribution[{0, 1}]]
```

Thus  $P(Y=1)=\theta$ .

Note that we are *completely ignorant* about the result of this fight. Our uncertainty is a *epistemic* one, not an aleatory one. For an explanation of types of uncertainty, see [2].

Furthermore, suppose that X and Y are independent.

---

## Bayesian Inference

Firstly, we have to develop a prior distribution for  $X \cap Y$ , i.e. both random variable “happen together”. Assuming independence, this amounts to  $X * Y$ .

One difficulty remains however: We have to select a *specific* value for  $\theta$ . Because we are totally ignorant about the result of the fight, we “integrate out” the parameter  $\theta$ , thus determining the *expected value of  $\theta$* .

The expected value of the continuous random variable  $\theta$  is defined as follows :

$$E[\theta] = \int_0^1 f(\theta) * \theta d\theta.$$

```
In[133]:= f = PDF[UniformDistribution[{0, 1}], ε]
```

```
Out[133]= {1, 0 ≤ ε ≤ 1  
0, True}
```

```
In[134]:= Etheta = Integrate[f * ε, {ε, 0, 1}]
```

```
Out[134]= 1/2
```

Thus  $E[\theta] = \frac{1}{2}$ .

Setting  $\theta = \frac{1}{2}$  we get for the pdf of Y:

$$\text{In[135]:= pdftheta} = \text{PDF}\left[Y /. \theta \rightarrow \frac{1}{2}, y\right]$$

$$\text{Out[135]= } \begin{cases} \frac{1}{2} & Y == 0 \text{ || } Y == 1 \\ 0 & \text{True} \end{cases}$$

$$\text{In[136]:= Ybayes} = Y /. \theta \rightarrow \text{Etheta}$$

$$\text{Out[136]= BernoulliDistribution}\left[\frac{1}{2}\right]$$

Note that is exactly the same as for X. There is no difference in the modelling of aleatory and epistemic uncertainty, if you do a standard Bayesian inference.

### ■ The Bayesian Prior

We are now looking for  $P(X \cap Y)$ :

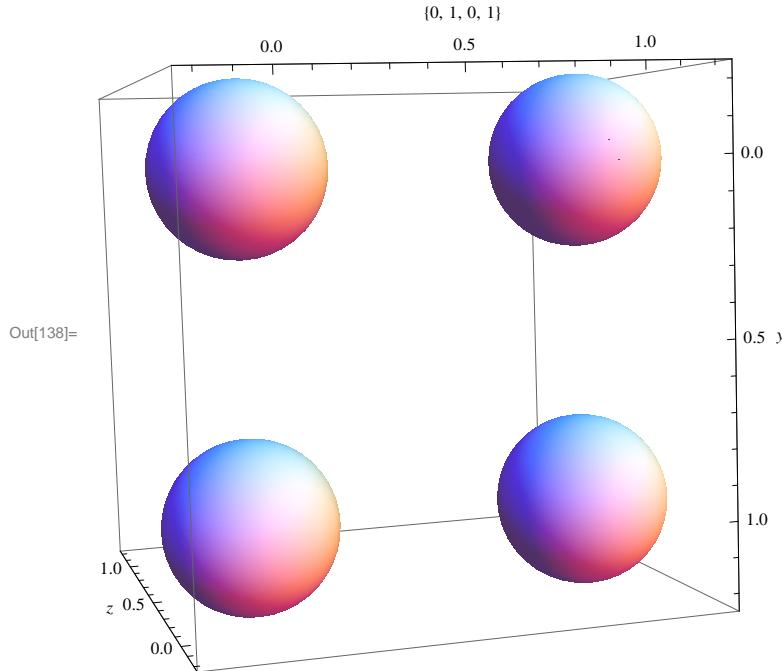
$$\text{In[137]:= priorBayes} = \text{PDF}[X, \text{xval} = \{0, 1, 1, 0\}] * \text{PDF}[Ybayes, \text{yval} = \{0, 1, 0, 1\}]$$

$$\text{Out[137]= } \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

Note that there are only four variations of (x,y).

Let's graph this prior mass distribution:

$$\text{In[138]:= Graphics3D[Table[Sphere[\{x, y, z1 = PDF[X, x] * PDF[Ybayes, y]\}, z1], \{x, 0, 1\}, \{y, 0, 1\}], PlotRange \rightarrow \{\{-0.25, 1.25\}, \{-0.25, 1.25\}, \{-0.25, 1.25\}\}, Axes \rightarrow \text{True}, AxesLabel \rightarrow \{x, y, z\}]}$$



### ■ The Bayesian Posterior

We are computing  $P(X^*Y | X=Y)$ , i.e. the posterior pdf. We are conditioning on the cases  $X=Y=0$  and  $X=Y=1$ . Note that we are dealing with the discrete case here.

$$P(H|E) = \frac{P(E|H) * P(H)}{P(E)}$$

$P(H)$ :

In[139]:= **ph = priorBayes**

$$\text{Out[139]}= \left\{ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right\}$$

P(E|H):

In[140]:= **peh = ph \* {1, 1, 0, 0}**

$$\text{Out[140]}= \left\{ \frac{1}{4}, \frac{1}{4}, 0, 0 \right\}$$

P(E):

In[141]:= **pe = Total [ph \* peh]**

$$\text{Out[141]}= \frac{1}{8}$$

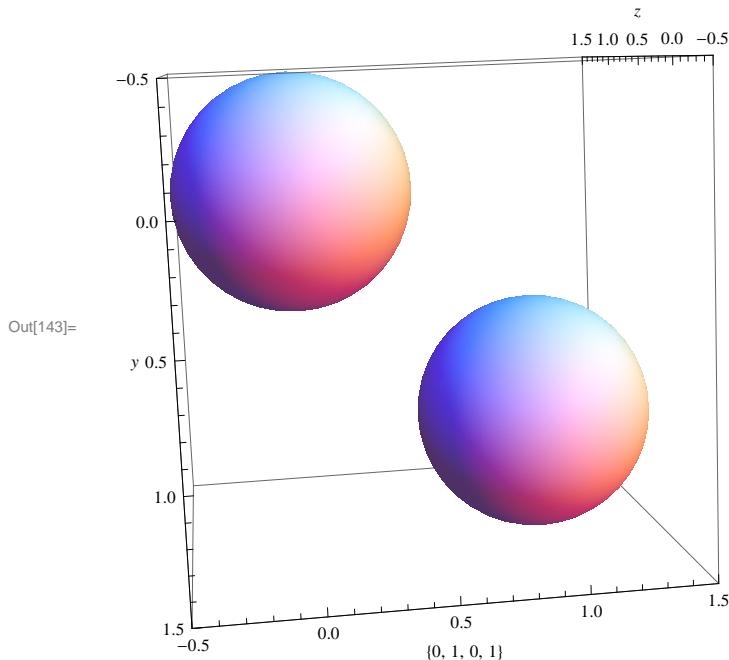
P(H|E):

In[142]:= **phe =  $\frac{\text{peh} * \text{ph}}{\text{pe}}$**

$$\text{Out[142]}= \left\{ \frac{1}{2}, \frac{1}{2}, 0, 0 \right\}$$

Let's graph this posterior mass distribution:

In[143]:= **Graphics3D[**  
 $\{ \text{Sphere}[\{0, 0, \text{phe}[[1]]\}, \text{phe}[[1]]], \text{Sphere}[\{1, 1, \text{phe}[[2]]\}, \text{phe}[[2]]] \},$   
 $\text{PlotRange} \rightarrow \{ \{-0.5, 1.5\}, \{-0.5, 1.5\}, \{-0.5, 1.5\} \},$   
 $\text{Axes} \rightarrow \text{True}, \text{AxesLabel} \rightarrow \{x, y, z\} \bigr]$



## Robust Bayesian Inference

There is a second possibility of handling this problem: Don't integrate off  $\theta$ . Just compute the posterior, given  $\theta$ , i.e.  $P(X^*Y|x=y)$ .

```
In[144]:= Clear[\theta]; Y = BernoulliDistribution[\theta]
Out[144]= BernoulliDistribution[\theta]
```

### ■ The robust Bayesian Prior

Again, we are looking at  $P(X \cap Y)$ :

$P(H)$ :

```
In[145]:= priorBayes = PDF[X, x = {0, 1, 1, 0}] * PDF[Y, x = {0, 1, 0, 1}]
Out[145]=  $\left\{ \frac{1-\theta}{2}, \frac{\theta}{2}, \frac{1-\theta}{2}, \frac{\theta}{2} \right\}$ 
```

### ■ The robust Bayesian Posterior

```
In[146]:= ph = priorBayes
```

```
Out[146]=  $\left\{ \frac{1-\theta}{2}, \frac{\theta}{2}, \frac{1-\theta}{2}, \frac{\theta}{2} \right\}$ 
```

Compute the likelihood  $P(E|H)$ :

```
In[147]:= peh =  $\left\{ \frac{1}{2}, \frac{1}{2}, 0, 0 \right\}$ 
```

```
Out[147]=  $\left\{ \frac{1}{2}, \frac{1}{2}, 0, 0 \right\}$ 
```

$P(E)$ :

```
In[148]:= pe = Total[ph * peh]
```

```
Out[148]=  $\frac{1-\theta}{4} + \frac{\theta}{4}$ 
```

$P(H|E)$ :

```
In[149]:= phe =  $\frac{peh * ph}{pe}$  // FullSimplify
```

```
Out[149]=  $\{1-\theta, \theta, 0, 0\}$ 
```

Now everything depends on the parameter  $\theta$ . Our knowledge about the aleatory random variable X is gone. Only the epistemic uncertainty remains.

## Bibliography

1. Gelman, Andrew, *The Boxer, the Wrestler, and the Coin Flip*. The American Statistician, 2006.
2. O'Hagan, Tony, *Dicing with the unknown*. significance, 2004.